Calc III Review

Vectors:

Vector notation: $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

magnitude: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

$c \cdot \vec{u} = \langle c \cdot u_1, c \cdot u_2, c \cdot u_3 \rangle$

Unit Vector: $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$, $\vec{v} = |\vec{v}|\,\hat{v}$

Distance: $x_2 - x_1$ (1D)

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ (2D)

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2(z_1-z_2)^2}$ (3D)

Dot Product: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$ (scalar)

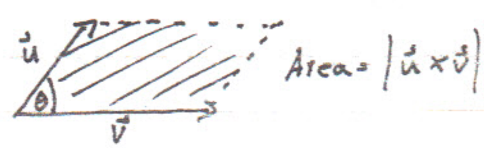
if $\vec{u} \perp \vec{v}$ then $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\left(\frac{\pi}{2}\right) = 0$

Projections: the component of $\vec{u}$ in the direction of $\vec{v}$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\left(\hat{v}\right) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

$$\text{scal}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \left|\text{proj}_{\vec{v}}\vec{u}\right|$$

Cross Product: $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta =$



$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\imath}(u_2v_3 - u_3v_2) - \hat{\jmath}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1)$$

if $\vec{u} \parallel \vec{v}$ then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(0) = 0$

$\vec{u} \times \vec{v} = \vec{0}$

Vector functions: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Point and // vector: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle v_1, v_2, v_3 \rangle$$

$$x = x_0 + tv_1 \quad , \quad y = y_0 + tv_2 \quad , \quad z = z_0 + tv_3$$

Vector Calculus: $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Unit Tangent Vector: $\hat{r}'(t) = \dfrac{\vec{r}'(t)}{|\vec{r}(t)|}$

Arc Length: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$L = \int_a^b |\vec{r}'(t)|\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

Reperamitrization: ex.) $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle \quad , \quad 1 \le t \le 2$

if $t = e^s$......

$$\vec{r}_2(s) = \langle e^s, e^{2s}, e^{3s} \rangle, \quad 0 \le s \le \ln 2$$

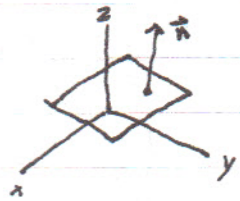
$$\therefore \vec{r}_1(t) = \vec{r}_2(s)$$

Calc III Review

Planes and Surfaces

Multivariable functions: $z = f(x,y)$, $w = f(x,y,z)$

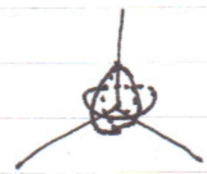
Planes: Defined by 3 points
or
1 point and a normal vector $(\vec{n})$



$\vec{n} = \langle a, b, c \rangle$, $P_0(x_0, y_0, z_0)$
$a(x-x_0) + b(y-y_0) + c(z-z_0) = d$, d is constant

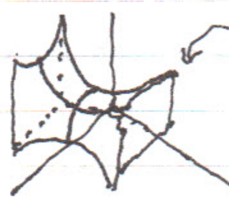
Quadratic Surface: function of 3 variables

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
↳ 3 elliptical traces



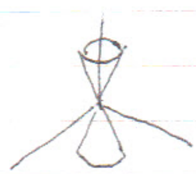
Elliptical paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
↳ 2 parabolic traces
↳ 1 elliptical traces

Hyperbolic paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
↳ 2 parabolic traces
↳ 1 hyperbolic trace

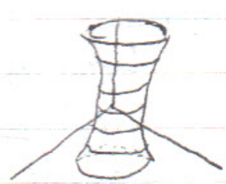


saddle

Elliptical cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
↳ 2 hyperbolic traces
↳ 1 elliptical trace



Hyperboloid of 1 sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
↳ 2 hyperbolic traces
↳ 1 elliptical traces



Hyperboloid of 2 sheets: $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

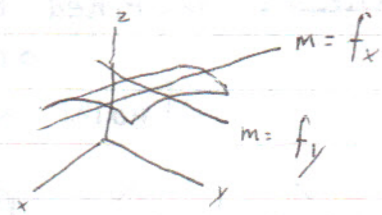
2 hyperbolic traces
1 elliptical trace

Partial Derivatives: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

ex) $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$



Chain Rule: $z(t) = f(x(t), y(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right)$$

Directional Derivatives: $D_{\hat{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$

$\hat{u} = \langle a, b \rangle$

Gradient Vector: $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = |\vec{\nabla} f| \cos\theta$

gradient vector gives vector of maximum change (highest slope)

Tangent Plane: $z = f(x,y)$, $F(x,y,z) = f(x,y) - z = 0$

$z = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + f(x_0,y_0)$

Differentials: $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

Max/Min: $z = f(x,y)$ @ (a,b)

$f_x = 0$, $f_y = 0$ to find critical points

Second Derivative Test: $D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

Local min if $D > 0$ and $f_{xx}(a,b) > 0$
Local max if $D > 0$ and $f_{xx}(a,b) < 0$
Saddle point: $D < 0$

Lagrange Multipliers: $\vec{\nabla} f = \lambda \vec{\nabla} g$, λ is a constant

Calc III Review

Double Integrals

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Average Value function

$$f_{avg} = \frac{1}{Area(R)} \iint_R f(x,y)\,dA$$

Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$V = \iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Triple Integrals

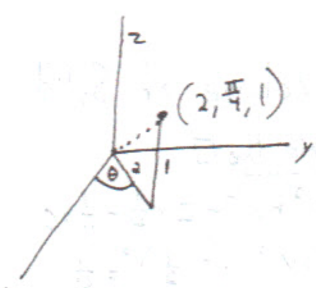
$$\iiint_V f(x,y,z)\,dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$ [order can also be switched]

$$\int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Cylindrical Coordinates

$f(x,y,z) \rightarrow f(r,\theta,z)$

ex.) $(r,\theta,z) = (2, \frac{\pi}{4}, 1)$



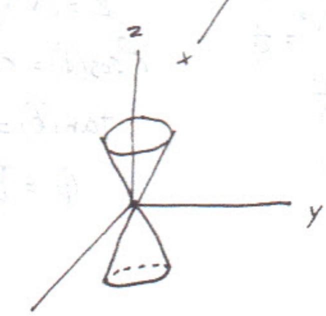
$x = r\cos\theta$

$y = r\sin\theta$ $\quad z = z$

$r^2 = x^2 + y^2$

ex.) Circular cone: $z^2 = x^2 + y^2$

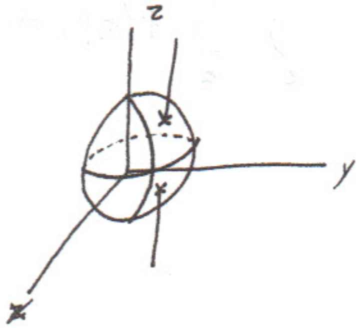
$z^2 = r^2$



Cylindrical Coodinates (continued)

$$\int_a^b \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

ex.) Sphere



$x^2+y^2+z^2=1$

$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$-1 \le x \le 1$

$r^2+z^2=1$

$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

Spherical Coordinates

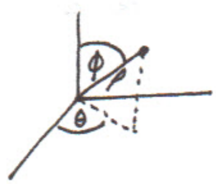
ρ = distance from origin

ϕ = angle from z-axis

θ = angle from x-axis

(ρ, ϕ, θ)

$\rho = \sqrt{x^2+y^2+z^2}$



$\rho = C$ Sphere

$\phi = C$ Cone

$\theta = C$ Plane

$x = \rho\sin(\phi)\cos(\theta)$

$y = \rho\sin(\phi)\sin(\theta)$

$z = \rho\cos(\phi)$

$r = \rho\sin\phi$

$z = \rho\cos\phi$

$\rho = \sqrt{x^2+y^2+z^2}$

$$\iiint_V f(x,y,z)\, dV = \iiint f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2\sin(\phi)\, d\rho\, d\phi\, d$$

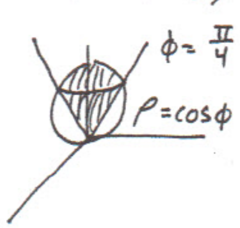
$$= \iiint f\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

ex.) Volume of Solid above $z=\sqrt{x^2+y^2}$ and below $z = x^2+y^2+z^2$

complete the □

$x^2+y^2+z^2-z+\frac{1}{4}-\frac{1}{4}=0$

$x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$



cone

$z=\sqrt{x^2+y^2}$

$\rho\cos(\phi) = r = \rho\sin(\phi)$

$\tan(\phi)=1$

$\phi = \frac{\pi}{4}$

sphere

$x^2+y^2+z^2=z$

$\rho^2 = \rho\cos(\phi)$

$\rho = \cos(\phi)$

boundaries

$0 \le \rho \le \cos(\phi)$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos(\phi)} \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$\boxed{V = \frac{\pi}{8}}$$

Calc III Review

Vector Calc

Vector Field

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

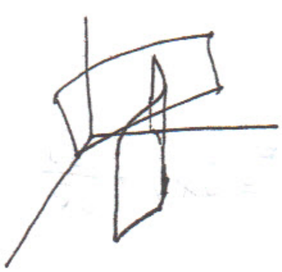
Gradient Field

$$\vec{\nabla} f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle$$

Potential Function

$\vec{F}(x,y,z) = \vec{\nabla}\phi(x,y,z)$, where ϕ is the potential function

Line Integral ..... of Vector Fields



$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(x(t), y(t), z(t)\big) \cdot \vec{r}\,'(t)\, dt$$

Circulation

$\int_C \vec{F} \cdot \hat{T}\, ds$, $\hat{T}$ is the unit tangent vector

Flux

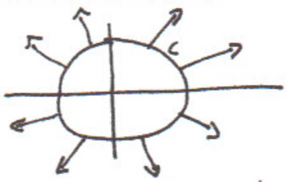
$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \left[f \cdot \frac{dy}{dt} - g \cdot \frac{dx}{dt} \right] dt,$$

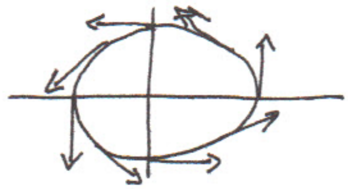
$\vec{n}$ is the normal vector

$\vec{F} = \langle f, g \rangle$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

Flux and Circulation (continued)





On unit Circle....

$\vec{F} = \langle x, y \rangle$

zero circulation

positive flux

$\vec{F} = \langle -y, x \rangle$

positive circulation

zero flux

Conservative Vector Fields

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ if $\vec{F} = \langle P, Q \rangle$ and defined on an open, connected region

Conservative

Fundimental Theorem of Line Integrals

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

To find the potential function.....ϕ

$$\vec{F} = \vec{\nabla}\phi$$

ex.) $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{z} \rangle = \vec{\nabla}\phi$

$\dfrac{\partial\phi}{\partial x} = y^2$, $\dfrac{\partial\phi}{\partial y} = 2xy + e^{3z}$, $\dfrac{\partial\phi}{\partial z} = 3ye^{z}$

$$\int \partial\phi = \int y^2\, \partial x$$

$$\phi = xy^2 + g(y, z)$$

$$\frac{\partial\phi}{\partial y} = 2xy + \frac{\partial g}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = 2xy + e^{3z} = 2xy + \frac{\partial g}{\partial y}$$

$$\frac{\partial g}{\partial y} = e^{3z}$$

$$\int \partial g = \int e^{3z}\, dy$$

$$g = ye^{3z} + h(z)$$

$$\phi = x^2y^2 + ye^{3z} + h(z)$$

$$\frac{\partial\phi}{\partial z} = 3ye^{3z} + \frac{\partial h}{\partial z} = 3ye^{3z} = \frac{\partial\phi}{\partial z}$$

$$\frac{\partial h}{\partial z} = 0 \longrightarrow h(z) = C$$

$$\boxed{\phi(x, y, z) = xy^2 + ye^{3z} + C}$$

Calc III Review

Green's Theorem (Circulation Form)

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$$

↳ Flux Form

$$\oint_C P\,dy - Q\,dx = \iint_D \left(\frac{dP}{dx} + \frac{dQ}{dy}\right) dA$$

Curl

$$\text{curl}\,\vec{F} \rightarrow \vec{\nabla} \times \vec{F} = \left\langle \frac{dR}{dy} - \frac{dQ}{dz}, \frac{dP}{dz} - \frac{dR}{dx}, \frac{dQ}{dx} - \frac{dP}{dy} \right\rangle$$

Divergence

$$\text{div}\,\vec{F} \rightarrow \vec{\nabla} \cdot \vec{F} = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}$$

Surface Area

$$\iint_D \left| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right| dA$$

Special Condition: $z = g(x,y)$

$$= \iint_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\, dA$$

Surface Integral

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \left| \frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv} \right| dA$$

Surface Integral of a Vector Field

$$\iint_S \vec{F} \cdot \hat{n}\,dS = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\,dA = \iint_R (-f z_x - g z_y + h)\,dA$$

Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dS$$

Divergence Theorem

$$\iint_S \vec{F} \cdot ds = \iiint_E \vec{\nabla} \cdot \vec{F} \, dV$$