Differential Equations Review

First Order

Separation of Variables:

$$\frac{dx}{dt} = f(x)\cdot g(t)$$

$$\int \frac{dx}{f(x)} = \int g(t)\,dt$$

$$F(x) = G(t) + C$$

Linear Equations:

$$a_1(t)\frac{dx}{dt} + a_0(t)\,x = E(t)$$

$E(t) = 0$: homogenous O.D.E.

$$\frac{dx}{dt} + r(t)\,x = q(t)$$

① $\frac{dx}{dt} + r(t)\,x = 0$ set $q(t)$ equal to 0

② $x = K\,h(t)$ solve for $x(t)$

③ $x = K(t)\,h(t)$ make constant K a function of t

④ $\frac{dK}{dt} = \frac{q(t)}{h(t)}$ set $\frac{dK}{dt}$ equal to $q(t)$ over $h(t)$

⑤ $K(t) = \int \frac{q(t)}{h(t)}\,dt$ Solve for $K(t)$

⑥ $x(t) = K(t)\cdot h(t)$ fill $K(t)$ and $h(t)$ back into $x(t)$

Normality

- all $a(t)$ must be continuous
- $a_n(t) \neq 0$ (highest order $a(t)$)

Existance: (Must be Normal to apply)

$$\frac{dx}{dt} = f(x,t)$$

$\frac{\partial f}{\partial x}$ → must be continuous

ex.) $t\cdot\frac{dx}{dt} = 2x+1$

$$\frac{dx}{dt} = \frac{2x+1}{t}$$

$\frac{\partial f}{\partial x} = \frac{2}{t}$, not continuous @ $t=0$

ex.) $(t^2-1)\frac{dx}{dt} = 2xt$

$$\frac{dx}{dt} = \frac{2xt}{t^2-1}$$

$$\frac{\partial f}{\partial x} = \frac{2t}{t^2-1}$$

$$t^2-1 \neq 0$$

$$t \neq \pm 1$$

$t: (-\infty,-1)(-1,1)(1,\infty)$

$$\int \frac{dx}{x} = \frac{2t}{t^2-1}\,dt$$

$$x(t) = K(t^2-1)$$

$$x(\pm 1) = K((\pm 1)^2-1)$$

$$x(\pm 1) = K\cdot 0$$

non-uniqueness

Matrices

Determinant:

$$\Delta = \det\begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = \begin{vmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{vmatrix} = b_{11}\begin{vmatrix} b_{22} & b_{32} \\ b_{23} & b_{33} \end{vmatrix} - b_{21}\begin{vmatrix} b_{12} & b_{32} \\ b_{13} & b_{33} \end{vmatrix} + b_{31}\begin{vmatrix} b_{12} & b_{22} \\ b_{13} & b_{23} \end{vmatrix}$$

$$b_{11}(b_{22}b_{33} - b_{32}b_{23}) - b_{21}(b_{12}b_{33} - b_{32}b_{13}) + b_{31}(b_{12}b_{23} - b_{22}b_{13})$$

Cramer's Determinant Test:

If $\Delta = 0$

- ↳ no solution

  or

- ↳ infinitely many solutions

If $\Delta \neq 0$

- ↳ has a unique solution
- ↳ **Cramer's Rule**

$$C_j = \begin{vmatrix} b_{11} & \cdots & \overset{j}{r_1} & \cdots & b_{1n} \\ \vdots & & \vdots & & \\ b_{n1} & \cdots & r_n & \cdots & b_{nn} \end{vmatrix} \div \Delta$$

① replace column j with constants on right side of equation (r)

② divide by the determinant (Δ)

Row Reduction:

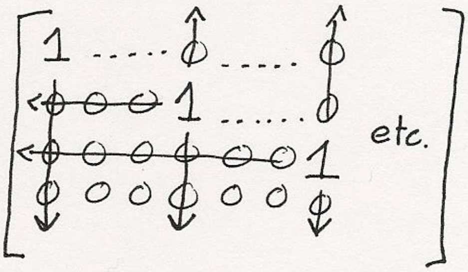
Row Operations

① Adding Multiples of rows

② Multiply by a constant

③ Switch rows

try to get:



ex.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 1 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & 3 & 0 & 3 \end{array}\right] \xrightarrow[R_2 \to \frac{R_2}{-3}]{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + z = 2, \quad \left.\begin{array}{l} x = 2 - a \\ y = 1 \\ z = a \end{array}\right\} \vec{x} = \begin{bmatrix} 2-a \\ 1 \\ a \end{bmatrix}, \quad \boxed{\vec{x} = a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

Differential Equations Review

Nth Order

$h(t) =$ general solution

$p(t) =$ particular solution

$$D^n x = \frac{d^n x}{dt^n}$$

$L =$ linear differential operator

$\hookrightarrow L = a_n(t) D^n + \dots + a_1(t) D + a_0(t)$

$\hookrightarrow L(x_1 + x_2) = Lx_1 + Lx_2$

Linear	Homogeneous
$Lx = E(t)$	$Lx = E(t) = 0$

ex.) $D^2x - Dx - 2x = -t + 4$, $p(t) = At + B$, $H(t) = c_1 e^{2t} + c_2 e^{-t}$

$D^2p - Dp - 2p = -t + 4$ $\quad Dp = A$

$0 - A - 2(At + B) = -t + 4$ $\quad D^2p = 0$

$-2A = -1$, $-2B - A = 4$

$A = \frac{1}{2}$, $B = \frac{-9}{4}$

$x(t) = H(t) + p(t)$

$$\boxed{x(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{2}t - \frac{9}{4}}$$

Wronskian:

$$W[h_1, \dots, h_n](t_0) = \begin{vmatrix} h_1(t_0) & h_2(t_0) & \dots & h_n(t_0) \\ h_1'(t_0) & h_2'(t_0) & \dots & h_n'(t_0) \\ \vdots & \vdots & & \vdots \\ h_1^{(n-1)}(t_0) & h_2^{(n-1)}(t_0) & \dots & h_n^{(n-1)}(t_0) \end{vmatrix}$$

If $W[h_1, \dots, h_n](t_0) \neq 0$

$\hookrightarrow Lx = 0$ is $x = c_1 h_1(t) + \dots + c_n h_n(t)$

ex.)

$$\left.\begin{matrix} h_1(t) = 1, & h_2(t) = t^2, & h_3(t) = \frac{1}{t} \\ h_1'(t) = 0, & h_2'(t) = 2t, & h_3'(t) = \frac{-1}{t^2} \\ h_1''(t) = 0, & h_2''(t) = 2, & h_3''(t) = \frac{2}{t^3} \end{matrix}\right\} W[h_1, h_2, h_3](1) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 1(4 + 2) = \boxed{6 \neq 0} \quad \boxed{\text{linearly independent}}$$

Linear Dependency:

for $c_1 h_1(t) + \dots + c_n h_n(t) = 0$

linearly dependent	linearly independent
$\hookrightarrow c_i \neq 0$	$\hookrightarrow c_1 = c_2 = \dots = c_n = 0$
(at least one of the c's does not equal zero)	(all c's equal zero)
	or...
	$\hookrightarrow W[h_1, \dots, h_n](t_0) \neq 0$

Homogeneous Linear Equations:

Real Roots:

① $(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)x = 0$ homogeneous equation

② $p(r) = a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$ set characteristic polynomial $= 0$

③ $r = ?$, $\lambda = r = ?$ solve for "r" which is "λ"

④ $h(t) = e^{\lambda t}$ plug into homogeneous equation

Complex Root:

③ $\lambda = \alpha \pm \beta i$

④ $h_1(t) = e^{\alpha t}\cos(\beta t)$, $h_2(t) = e^{\alpha t}\sin(\beta t)$

Multiplicity:

③ $\lambda = ?$ mult. of k

④ $h(t) = e^{\lambda t}, te^{\lambda t}, \dots, t^{k-1}e^{\lambda t}$

Exponential Shift:

$$L(D)e^{\lambda t}x = e^{\lambda t}\cdot L(D+\lambda)x$$

Annihilator:

If $\phi(t)$ is the function,

$A(D)\phi = 0$ ($A(D)$ annihilates ϕ)

ex.) $(D^2-1)x = e^t$

$(D^2-1)h = 0$

$\lambda = \pm 1$

$h_1(t) = e^t$, $h_2(t) = e^{-t}$

$A(D) = D - 1$

$(D-1)(D^2-1)p = 0$

$(D-1)^2(D+1)p = 0$

$\lambda = -1, 1$ mult. 2

$$p(t) = \underbrace{\cancel{K_1 e^{-t}} + \cancel{K_2 e^{t}}}_{\text{already show up in } h(t)} + K_3 te^t$$

$p(t) = K_3 te^t$

$(D^2-1)K_3 te^t = E(t)$

exponential shift

$K_3 \underset{\lambda=1}{e^t}((D+1)^2-1)t = E(t)$

$D \to D+\lambda$, $D+1$

$K_3 e^t(D^2+2D)t = E(t)$

$K_3 e^t(0+2) = E(t)$

$2K_3 e^t = e^t$

$\to K_3 = \frac{1}{2}$

$p(t) = \frac{1}{2}te^t$

$$\boxed{x(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t}$$

Method of Annihilators:

(Undetermined Coeficients)

① $P(D)x = E(t)$ Set equation in this form

② $P(D)x = 0$ Solve homogeneous equation

③ $h(t) = ?$

④ $A(D)E(t) = 0$ Find annihilator for $E(t)$

⑤ $A(D)\cdot P(D)p = 0$ Multiply annihilator by $P(D)$ on "p" equals "zero" to solve for "p"

↖"p"

⑥ $p(t) =$ stuff ~~in h(t)~~ + other stuff drop terms that also appear in $h(t)$

$p(t) =$ other stuff

⑦ $P(D)p(t) = E(t)$ set $P(D)$ on $p(t)$ equal to $E(t)$

⑧ $K_1 = ?, K_2 = ?, \dots$ solve for undetermined coeficients

⑨ $p(t) = K\cdot$ other stuff plug back into $p(t)$

⑩ $x(t) = p(t) + h(t)$

Variation of Parameters:

① $\left(D^n + \dots + b_1(t)D + b_0(t)\right)x = E(t)$ Get equation in this form

② $\left(D^n + \dots + b_1(t)D + b_0(t)\right)h = 0$ Solve homogeneous equation

③ $h(t) = c_1 h_1(t) + \dots + c_n h_n(t)$

④ Solve system of equations

$$\begin{aligned} c_1'(t)h_1(t) + \dots + c_n'(t)h_n(t) &= 0 \\ c_1'(t)h_1'(t) + \dots + c_n'(t)h_n'(t) &= 0 \\ \vdots \qquad\qquad \vdots \qquad &= 0 \\ &= 0 \\ c_1'(t)h_1^{(n-1)}(t) + \dots + c_n'(t)h_n^{(n-1)}(t) &= E(t) \end{aligned}$$

⑤ $\int [c_1'(t), \dots, c_n'(t)]\,dt$ Integrate to get $c_1, \dots, c_n$

⑥ $p(t) = c_1(t)h_1(t) + \dots + c_n(t)h_n(t)$ Plug into particular Solution

⑦ $x(t) = p(t) + h(t)$

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ex.) $(4D^2 - 4D + 1)x = \frac{8}{t^2}e^{\frac{t}{2}}$

$x'' - x' + \frac{x}{4} = \frac{2}{t^2}e^{\frac{t}{2}}$ ①

$x'' - x' + \frac{x}{4} = 0$ ②

$r^2 - r + \frac{1}{4} = 0$

$(r - \frac{1}{2})^2 = 0$

$\lambda = \frac{1}{2}$ mult. 2

$h(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}$ ③

$h_1(t) = e^{\frac{t}{2}}, \quad h_2(t) = te^{\frac{t}{2}}$

$\swarrow \; p(t) = c_1(t)e^{\frac{t}{2}} + c_2(t)te^{\frac{t}{2}}$

$c_1' e^{\frac{t}{2}} + c_2' t e^{\frac{t}{2}} = 0$ ④

$c_1'(\frac{1}{2})e^{\frac{t}{2}} + c_2'(e^{\frac{t}{2}} + \frac{1}{2}te^{\frac{t}{2}}) = \frac{2}{t^2}e^{\frac{t}{2}}$

$c_1' + tc_2' = 0, \quad \frac{1}{2}c_1' + (1 + \frac{1}{2}t)c_2' = \frac{2}{t^2}$

$c_1' = \frac{-2}{t}$

$c_2' = \frac{2}{\ }$

$\longrightarrow c_1(t) = \int c_1'(t)\,dt = \int \left(\frac{-2}{t}\right)dt$ ⑤

$c_1(t) = -2\text{Ln}(t) + K_1$

$c_2(t) = \int c_2'(t)\,dt = \int \frac{2}{t^2}\,dt$

$c_2(t) = \frac{-2}{t} + K_2$

If $K_2 + K_1 = 0$ .....

$p(t) = c_1(t)e^{\frac{t}{2}} + c_2(t)te^{\frac{t}{2}}$

$p(t) = (-2\text{Ln}(t))e^{\frac{t}{2}} + \left(\frac{-2}{t}\right)te^{\frac{t}{2}}$

$p(t) = -2\text{Ln}(t)e^{\frac{t}{2}} + -2e^{\frac{t}{2}}$ ⑥

$x(t) = \underbrace{c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}}_{h(t)} \underbrace{- 2e^{\frac{t}{2}}\text{Ln}(t) - 2e^{\frac{t}{2}}}_{p(t)}$ ⑦

$x(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}} - 2e^{\frac{t}{2}}\text{Ln}(t) - \cancel{2e^{\frac{t}{2}}}$ redundent since $e^{\frac{t}{2}}$ is in $h(t)$

$$\boxed{x(t) = e^{\frac{t}{2}}\left(c_1 + tc_2 - 2\text{Ln}(t)\right)}$$

Laplace Transform

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Formulas:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{(n-1)!} t^{n-1}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{1}{(n-1)!} t^{n-1} \cdot e^{at}$$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin(at)$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

First Differentiation:

$$\mathcal{L}[D^n x] = s^n \mathcal{L}[x] - s^{n-1}x(0) - s^{n-2}x'(0) - \dots - x^{(n-1)}(0)$$

First Shift:

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \qquad \mathcal{L}^{-1}[F(s)] = e^{at}\mathcal{L}^{-1}[F(s+a)]$$

Second Shift:

$$\mathcal{L}[u_a(t)g(t)] = e^{-as}\mathcal{L}[g(t+a)] \qquad \mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$$

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \qquad (1-u_a(t))f(t) = \begin{cases} f(t), & t < a \\ 0, & t \geq a \end{cases}$$

Convolution:

$$(f \star g)(t) = \int_0^t f(t-u)\,g(u)\,du \qquad \mathcal{L}[(f \star g)(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

$$\sin(\alpha t) \star \cos(\alpha t) = \frac{t}{2}\sin(\alpha t)$$

$$\sin(\alpha t) \star \sin(\alpha t) = \frac{1}{2\alpha}\sin(\alpha t) - \frac{t}{2}\cos(\alpha t)$$

$$\mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F(s)] \star \mathcal{L}^{-1}[G(s)]$$

Differential Equations Review

Laplace Transform Examples:

First Differentiation

$$(D^3 - 4D)x = 0 \quad , x(0)=4, \; x'(0) = x''(0) = 8$$

$$\mathcal{L}[D^3x - 4Dx] = \mathcal{L}[0]$$

$$\mathcal{L}[D^3x] - 4\mathcal{L}[Dx] = 0$$

$$(s^3\mathcal{L}[x] - s^2x(0) - s^1x'(0) - s^0x''(0)) - 4(s\mathcal{L}[x] - x(0)) = 0$$

$$s^3\mathcal{L}[x] - 4s^2 - 8s - 8 - 4s\mathcal{L}[x] + 16 = 0$$

$$\mathcal{L}[x](s^3 - 4s) = 4s^2 + 8s - 8$$

$$\mathcal{L}[x] = 4\frac{(s^2+2s-2)}{s(s+2)(s-2)} \longrightarrow \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-2}$$

$$A(s+2)(s-2) + B(s)(s-2) + C(s)(s+2) = s^2 + 2s - 2$$

$$A = \frac{1}{2} \qquad B = \frac{-1}{4} \qquad C = \frac{3}{4}$$

$$\mathcal{L}[x] = \frac{2}{s} - \frac{1}{s+2} + \frac{3}{s-2}$$

$$x(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$\boxed{x(t) = 2 - e^{-2t} + 3e^{2t}}$$

Second Shift

$$\mathcal{L}^{-1}\left[\frac{se^{-5s}}{s^2+16}\right] = \mathcal{L}^{-1}\left[e^{-5s}\underbrace{\left(\frac{s}{s^2+4^2}\right)}_{F(s)}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4^2}\right] = \cos(4t) = f(t)$$

$$\mathcal{L}^{-1}\left[\frac{se^{-5s}}{s^2+16}\right] = u_5(t)\cos(4(t-5))$$

$$= \boxed{u_5(t)\cos(4t-20)}$$

Convolution

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+s-2}\right] = \mathcal{L}^{-1}\left[\left(\frac{1}{s-1}\right)\left(\frac{1}{s+2}\right)\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

$$= \underbrace{e^t}_{f(t)} * \underbrace{e^{-2t}}_{g(t)} = \int_0^t f(t-u)g(u)\,du$$

$$= \int_0^t e^{t-u}e^{-2u}\,du$$

$$= e^t\int_0^t e^{-3u}\,du$$

$$p = -3u$$

$$dp = -3du \rightarrow du = \frac{-1}{3}dp$$

$$\rightarrow = e^t\left(\frac{-1}{3}\right)\int_0^{-3t} e^p\,dp$$

$$= \frac{-1}{3}e^t\left(e^p\Big|_0^{-3t}\right)$$

$$= \frac{-1}{3}e^t\left(e^{-3t} - 1\right)$$

$$\boxed{= \frac{1}{3}\left(e^t - e^{-2t}\right)}$$

Linear Systems

System of order "n":

$$D\vec{x} = A\vec{x} + \vec{E}(t)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \vec{E}(t) = \begin{bmatrix} E_1(t) \\ \vdots \\ E_n(t) \end{bmatrix}$$

ex.) $x_1' = x_2$

$x_2' = 3x_1 + 2x_2 + 6 - 8e^{t}$

$D\vec{x} = A\vec{x} + \vec{E}(t)$

Existance and Uniqueness (again):

Does $D\vec{x} = A\vec{x} + \vec{E}(t)$ work? -> Has solutions

$W[\vec{h}_1, \ldots, \vec{h}_n](t_0) \neq 0$ -> can generate general solution

ex.) $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \vec{h}_1(t) = \begin{bmatrix} 2e^{-2t} \\ -e^{-2t} \end{bmatrix}, \vec{h}_2(t) = \begin{bmatrix} e^{-T} \\ -e^{-t} \end{bmatrix}$

$D\vec{h}_1(t) = \begin{bmatrix} -4e^{-2t} \\ 2e^{-2t} \end{bmatrix} \qquad D\vec{h}_2(t) = \begin{bmatrix} -e^{-T} \\ e^{-t} \end{bmatrix}$

$A\vec{h}_1(t) = \begin{bmatrix} -6e^{-2t} + 2e^{-2t} \\ 2e^{-T} + 0 \end{bmatrix} = \begin{bmatrix} -4e^{-2t} \\ 2e^{-2t} \end{bmatrix}, \quad A\vec{h}_2(t) = \begin{bmatrix} -3e^{-T} + 2e^{-T} \\ e^{-t} + 0 \end{bmatrix} = \begin{bmatrix} -e^{-T} \\ e^{-t} \end{bmatrix}$

$$\boxed{\begin{array}{ll} D\vec{h}_1 = A\vec{h}_1 \ \checkmark & \text{Both are} \\ D\vec{h}_2 = A\vec{h}_2 \ \checkmark & \text{solutions} \end{array}}$$

$W[\vec{h}_1, \vec{h}_2](0) = \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = \boxed{-1 \neq 0 \quad \text{Generates general solution}}$

Linearly Independent:

$c_1\vec{V}_1 + \ldots + c_k\vec{V}_k = \vec{0}$

if at least one $c \neq 0$

ex.) $D\vec{x} = A\vec{x}$

$A = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{bmatrix} 3e^{4t} \\ e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-4t} \\ 3e^{-4t} \end{bmatrix}$

$\vec{h}_1(t) = \begin{bmatrix} 3e^{4t} \\ e^{4t} \end{bmatrix}, \quad D\vec{h}_1(t) = \begin{bmatrix} 12e^{4t} \\ 4e^{4t} \end{bmatrix}, \quad A\vec{h}_1(t) = \begin{bmatrix} (15-3)e^{4t} \\ (9-5)e^{4t} \end{bmatrix}$

$\vec{h}_2(t) = \begin{bmatrix} e^{-4t} \\ 3e^{-4t} \end{bmatrix}, \quad D\vec{h}_2(t) = \begin{bmatrix} -4e^{-4t} \\ -12e^{-4t} \end{bmatrix}, \quad A\vec{h}_2(t) = \begin{bmatrix} (5-9)e^{-4t} \\ (3-15)e^{-4t} \end{bmatrix}$

Eigen Values:

① $\det(A - \lambda I) = 0$

② solve for λ

Eigen Vectors:

③ choose λ

④ $(A - \lambda I)\vec{v} = \vec{0}$

⑤ Solve for $\vec{v}$

⑥ Repeat for other λ's

ex.) $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

① $\det(A - \lambda I) = 0 = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) - 1 = 0$

$= 9 - 6\lambda + \lambda^2 - 1$

$\lambda^2 - 6\lambda + 8 = 0$

② $(\lambda - 2)(\lambda - 4) = 0$

$\boxed{\lambda = 2, 4}$

③ $\lambda = 2$

④ $(A - 2I)\vec{v}_1 = \vec{0}$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} v_1 - v_2 = 0 \\ v_1 = v_2 \end{array}$ -> pick $v_2 = 1$ ⑤ $\boxed{\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

⑥ $\lambda = 4$

$(A - 4I)\vec{V}_2 = \vec{0}$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} v_1 + v_2 = 0 \\ v_1 = -v_2 \end{array}$ -> pick $v_2 = -1$ $\boxed{\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

Differential Equations Review

Homogenous Systems:

$$\vec{h}_1(t) = e^{\lambda_1 t} \cdot \vec{V}_1$$

$$\vdots$$

$$\vec{h}_n(t) = e^{\lambda_n t} \cdot \vec{V}_n$$

$$\vec{x} = c_1 \vec{h}_1 + \ldots\ldots + c_n \vec{h}_n$$

Complex Roots:

$$\lambda = \alpha \pm \beta i$$

$$e^{(\alpha + \beta i)} = e^{\alpha}\left(\cos(\beta) + i\sin(\beta)\right)$$

$$\vec{h}_1 = \text{Real}\left[\vec{h}_1\right]$$

$$\vec{h}_2 = \text{Imag}\left[\vec{h}_1\right]$$

ex.) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\det(A - \lambda I) = 0 = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}$

$$0 = (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$0 = (2-\lambda)(\lambda^2 - 2\lambda + 2)$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \boxed{\begin{array}{r} 1 \pm i = \lambda \\ 2 = \lambda \end{array}}$$

complex

$$\underline{\lambda = 1 + i}$$

$$(A - (1+i)I)\vec{V}_1 = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1-1-i & 0 & -1 & 0 \\ 0 & 2-1-i & 0 & 0 \\ 1 & 0 & 1-1-i & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & 1-i & 0 & 0 \\ 1 & 0 & -i & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} V_1 = i V_3 \\ V_2 = 0 \\ V_3 \text{ is free} \end{array}\right\} \rightarrow \begin{array}{l} \text{pick } v_3 = -i \\ v_1 = 1 \\ v_2 = 0 \end{array} \qquad \vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$$

$$\vec{h}(t) = e^{(1+i)t}\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = e^{t} \cdot e^{it}\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = e^{t}\left(\cos(t) + i\sin(t)\right)\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$$

$$\vec{h}(t) = e^{t}\begin{bmatrix} \cos(t) + i\sin(t) \\ 0 \\ \sin(t) - i\cos(t) \end{bmatrix}$$

$$\boxed{\begin{array}{l} \text{Real}\left[\vec{h}(t)\right] = \vec{h}_1(t) = \begin{bmatrix} \cos(t) \\ 0 \\ \sin(t) \end{bmatrix} e^{t} \\ \text{Imag}\left[\vec{h}(t)\right] = \vec{h}_2(t) = \begin{bmatrix} \sin(t) \\ 0 \\ -\cos(t) \end{bmatrix} e^{t} \end{array}}$$

continue for $\underline{\lambda = 2}$
to get $\vec{h}_3(t)$

$$\vec{x}(t) = C_1 \vec{h}_1(t) + C_2 \vec{h}_2(t) + C_3 \vec{h}_3$$

Multiplicity:

m = multiplicity #

$$(A - \lambda I)^m \vec{v} = \vec{0} \qquad \text{(use matrix multiplication)}$$

$$\vec{h}(t) = e^{\lambda t}\left[\vec{v} + t(A - \lambda I)\vec{v} + \frac{1}{2}t^2(A - \lambda I)^2\vec{v} + \ldots\ldots\right]$$

so.....

$$\vec{h}(t) = e^{\lambda t}\sum_{n=1}^{m}\left[\frac{1}{(n-1)!}t^{n-1}(A - \lambda I)^{n-1}\vec{v}\right]$$

Non-Homogeneous:

$$D\vec{x} = A\vec{x} + \vec{E}(t)$$

Variation of Parameters:

① $\vec{h}(t) = c_1\vec{h}_1(t) + \ldots + c_n\vec{h}_n(t)$

② $K_1'\vec{h}_1 + \ldots + K_n'\vec{h}_n = \vec{E}$

③ Solve for $K_1', \ldots, K_n'$

④ $\int [K_1', \ldots, K_n']\,dt$

⑤ $\vec{p}(t) = K_1 \cdot \vec{h}_1(t) + \ldots + K_n \cdot \vec{h}_n(t)$

⑥ $\vec{x}(t) = \vec{h}(t) + \vec{p}(t)$

ex.) $D\vec{x} = A\vec{x}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$0 = -\lambda\begin{vmatrix} -\lambda & 1 \\ -3 & 3-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix}$$

$$0 = -\lambda((-\lambda)(3-\lambda)+3)+1$$

$$0 = -(\lambda-1)^3, \quad \lambda = 1 \text{ mult. } 3$$

Multiplicity

$\lambda = 1$ mult. 3

$$(A-I)^3\vec{v} = \vec{0}$$

$$(A-I)^3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$(A-I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{h}_1(t) = e^t\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t \end{bmatrix}$$

$$\vec{h}_2(t) = e^t\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} te^t - t^2e^t \\ e^t - te^t - t^2e^t \\ -3te^t - t^2e^t \end{bmatrix}$$

$$\vec{h}_3(t) = e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t \\ e^t + 2te^t + \frac{1}{2}t^2e^t \end{bmatrix}$$

$$\vec{x}(t) = c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + c_3\vec{h}_3(t)$$

ex.) $D\vec{x} = A\vec{x} + \vec{E}(t)$

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}, \quad \vec{E}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Solve homogeneous

$$D\vec{h} = A\vec{h}, \quad \vec{h}_1 = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{h}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$K_1'(2e^t) + K_2'(e^{2t}) = e^t$$

$$K_1'(e^t) + K_2'(e^{2t}) = e^t$$

$$K_1' \cdot 2 + K_2' \cdot e^t = 1$$

$$K_1' \cdot 1 + K_2' \cdot e^t = 1$$

$$\Delta = \begin{vmatrix} 2 & e^t \\ 1 & e^t \end{vmatrix} = 2e^t - e^t = e^t$$

$$K_1' = \frac{\begin{vmatrix} 1 & e^t \\ 1 & e^t \end{vmatrix}}{\Delta} = 0$$

$$\int K_2'\,dt = C$$

(c goes to zero)

$$K_2' = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}{\Delta} = e^{-t}$$

$$\int K_2'\,dt = -e^{-t} + C$$

(c goes to zero)

$$p(t) = 0 - e^{-t}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$p(t) = \begin{bmatrix} -e^t \\ -e^t \end{bmatrix}$$

$$\boxed{\vec{x}(t) = c_1\begin{bmatrix} 2e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} -e^t \\ -e^t \end{bmatrix}}$$

Differential Equations Review

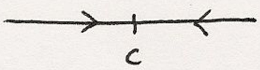
Qualitative Theory

Equilibria:

(x, y) where $\frac{dx}{dt} = \frac{dy}{dt} = 0$

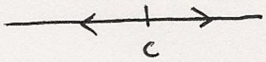
Phase Portrait:

stable { attractor:



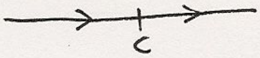
$\lim_{t \to \infty} f(x) = c$

unstable { repeller:

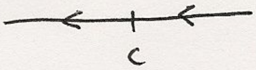


$\lim_{t \to \infty} f(x) = \pm\infty$

neither:



or



Integral Curve:

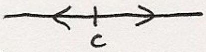
① find roots of A
↳ $\lambda = ?$

② find eigen vectors for λ

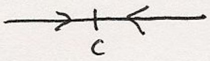
$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ ← describes a line

b = rise
a = run } $m = \frac{b}{a}$

③ If λ is ⊕



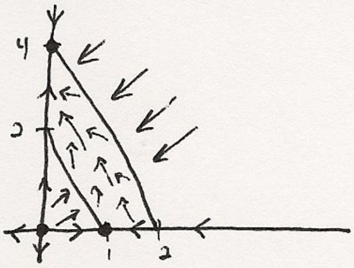
If λ is ⊖



ex.) $\frac{dx}{dt} = 2x - 2x^2 - xy = x(2 - 2x - y)$

$\frac{dy}{dt} = 4y - 2xy - y^2 = y(4 - 2x - y)$

equilibria: $(0,0), (1,0), (0,4)$



$\boxed{\text{Solutions approad } (1,0) \text{ and } (0,4)}$

ex.) $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$, $\det(A - \lambda I) = 0 = \begin{vmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix}$

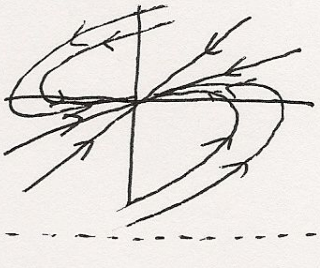
$\lambda = -2, -1$

$\underline{\lambda = -2}$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -3 & 3 & 0 \end{array}\right]$ $v_1 = v_2$ pick $v_2 = 1$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lim_{t \to -\infty} \vec{x} \Longrightarrow \lambda = -2$

$\underline{\lambda = -1}$

$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ -3 & 2 & 0 \end{array}\right]$ $3v_1 = 2v_2$ pick $v_2 = 3$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\lim_{t \to \infty} \vec{x} \to \lambda =$

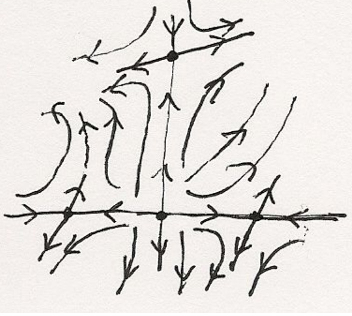


ex.) $f = \frac{dx}{dt} = xy - x^3 + x = x(y - x^2 + 1)$

$g = \frac{dy}{dt} = 3y + xy - y^2 = y(3 + x - y)$

$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y - 3x^2 + 1 & x \\ y & 3+x - 2y \end{bmatrix}$

equilibrium: $(0,3)$
$(1,0)$
$(-1,0)$
$(0,0)$



$(0,0)$: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \to$ $\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda = 3 \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ } unstable (repeller)

$(0,3)$: $\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix} \to$ $\lambda = -3 \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\lambda = 4 \rightsquigarrow \begin{bmatrix} 1 \\ 3/7 \end{bmatrix}$ } unstable (neither)

$(1,0)$: $\begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} \to$ $\lambda = -2 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda = 4 \rightsquigarrow \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

unstable (neither)

$(-1,0)$: $\begin{bmatrix} -2 & -1 \\ 0 & 2 \end{bmatrix} \to$ $\lambda = -2 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda = 2 \rightsquigarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Discriminant:

$$D(x,y) = E_{xx} \cdot E_{yy} - (E_{xy})^2$$

if $D > 0$ and $E_{xx} > 0$
↳ Stable (local min)

if $D > 0$ and $E_{xx} < 0$
↳ Stable (local max)

if $D < 0$
↳ unstable

Constant of Motion:

$$\frac{d}{dt} E(t) = 0$$

$$\frac{\partial E}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial E}{\partial y} \cdot \frac{dy}{dt} = 0$$

Lyapunov function:

$x' = f(x,y)$, $y' = g(x,y)$

$$\frac{d}{dt} E(x(t), y(t)) \leq 0$$

Critical point if $E_x = E_y = 0$

ex.) $\frac{dx}{dt} = -x^3 - x^2 - 2xy$ $\qquad E(x,y) = xe^{x-y^2}$

$\frac{dy}{dt} = 2x^3y - x - 1$

Verify Lyapunov:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} \leq 0$$

$$(1+x)e^{x-y}(-x^3-x^2-2xy) - 2xye^{x-y^2}(2x^3y-x-1)$$

$$-\left(\underset{\oplus}{x^2}\underset{\oplus}{(x-1)^2} + 4\underset{\oplus}{x^4}\underset{\oplus}{y^2}\right)e^{x-y^2} \leq 0 \quad ✓$$

Equilibria:

$$\left.\begin{array}{l} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{array}\right\} (-1,0)$$

Check if Critical:

$E_x(-1,0) = 0$

$E_y(-1,0) = 0$

classify:

$$D(x,y) = E_{xx} \cdot E_{yy} - (E_{xy})^2$$

$D(-1,0) = 2e^{-2} > 0$ -> Local extreme

$E_{xx}(-1,0) = e^{-1} > 0$ -> Local min

$$\boxed{\text{stable attractor}}$$